

METASTABILITY FOR SCALAR CONSERVATION LAWS IN A BOUNDED DOMAIN

CORRADO MASCIA¹, MARTA STRANI²

ABSTRACT. The initial-boundary-value problem for a viscous scalar conservation law in a bounded interval $I = (-\ell, \ell)$ is considered, with emphasis on *metastable dynamics*, whereby the time-dependent solution approaches its steady state in an asymptotically exponentially long time interval as the viscosity coefficient $\varepsilon > 0$ goes to zero. A rigorous analysis is used to analyze such slow motion for solutions belonging to a cylindrical neighborhood of a one-parameter family of approximate solutions $\{U^\varepsilon(\cdot; \xi)\}_{\xi \in I}$, to be considered as an approximate invariant manifold for the problem. The description consists in an ODE for the parameter ξ , describing the position of an internal transition layer, coupled with a PDE describing the evolution of the distance vector $v := u - U^\varepsilon(\cdot; \xi)$ from the manifold. By use of the properties of the linearized operator at U^ε , we estimate the size of both layer location ξ and distance vector v .

Key words. Metastability, slow motion, spectral analysis, viscous conservation laws.

1. INTRODUCTION

Metastability is a broad term describing the existence of a very sensitive equilibrium, possessing a weak form of stability/instability. Usually, such behavior is related to the presence of a small first eigenvalue for the linearized operator at the given equilibrium state, revealed at dynamical level by the appearance of slowly moving structures. Such circumstance comes into view in the analysis of different classes of evolutive PDEs, and it has been object of a wide amount of studies, covering many different areas. Among others, we emphasize the explorations on the Allen–Cahn equation, started in [5, 10], and the investigations on the Cahn–Hilliard equation, with the fundamental contributions [25, 1]. The analysis has been continued by many other scholars by means of a broad spectrum of techniques, and extended to a number of different models such as the Gierer–Meinhardt and Gray–Scott systems (see [29]), Keller–Segel chemotaxis system (see [9, 26]), general

¹Dipartimento di Matematica “G. Castelnuovo”, Sapienza – Università di Roma, P.le Aldo Moro, 2 - 00185 Roma (ITALY), AND Istituto per le Applicazioni del Calcolo, Consiglio Nazionale delle Ricerche (associated in the framework of the program “Intracellular Signalling”), mascia@mat.uniroma1.it

²Dipartimento di Matematica “G. Castelnuovo”, Sapienza – Università di Roma, P.le Aldo Moro, 2 - 00185 Roma (ITALY), Tel. +390649913406, E-mail address: strani@mat.uniroma1.it.

gradient flows (see [24]) and many others. The number of references is so vast that it would be impossible to mention all the contributions given in the area.

A pionereering article in the analysis of slow dynamics for evolutive parabolic partial differential equations has been authored by G. Kreiss and H.-O. Kreiss [14] and concerns with the scalar conservation law

$$(1.1) \quad \partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u,$$

with the space variable x belonging to a one-dimensional interval $I = (-\ell, \ell)$, $\ell > 0$. The primary prototype for the flux function f is given by the classical quadratic formula $f(s) = \frac{1}{2} s^2$, so that equation (1.1) becomes the so-called (*viscous*) *Burgers equation*. The parameter $\varepsilon > 0$ is considered as a strictly positive and small parameter. Equation (1.1) is complemented with Dirichlet boundary conditions

$$(1.2) \quad u(-\ell, t) = u_- \quad \text{and} \quad u(\ell, t) = u_+$$

for given data u^\pm to be discussed in details.

Burgers equation is considered as a (simplified) archetype of more complicate systems of partial differential equations arising in different fields of applied mathematics. Inspired by the equations of fluid-dynamics, the parameter ε is interpreted as a *viscosity coefficient* and the main problem is to identify and quantify its rôle in the emergence and/or disappearance of structure.

Formally, in the limit $\varepsilon \rightarrow 0^+$, the parabolic equation (1.1) reduces to a first-order quasi-linear equation of hyperbolic type

$$(1.3) \quad \partial_t u + \partial_x f(u) = 0,$$

whose standard setting is given by the *entropy formulation*, hence possessing possibly discontinuous solutions with speed of propagation s given by the **Rankine-Hugoniot relation**

$$(1.4) \quad s[[u]] = [[f(u)]]$$

(where $[[\cdot]]$ denotes the jump) together with appropriate **entropy conditions**. In addition, the treatment of the boundary conditions (1.2) is much more delicate with respect to the parabolic case, because of the eventual appearance of boundary layers.

Focusing on the more significant example, let us assume

$$(1.5) \quad f''(s) \geq c_0 > 0, \quad f'(u_+) < 0 < f'(u_-), \quad f(u_+) = f(u_-),$$

for some positive constant c_0 . The last two assumptions guarantee that the jump from the value u_- to u_+ is admissible and its speed of propagation, dictated by (1.4), is zero. In this case, equation (1.3) possesses a one-parameter family $\{U_{\text{hyp}}(\cdot; \xi)\}$ of stationary solutions satisfying the boundary conditions (1.2), given by

$$(1.6) \quad U_{\text{hyp}}(x; \xi) := u_- \chi_{(-\ell, \xi)}(x) + u_+ \chi_{(\xi, \ell)}(x)$$

where χ_I denotes the characteristic function of the set I . The dynamics determined by initial-value problem for (1.3)-(1.2) is very simple: it is possible to prove that every entropy solution converges *in finite time* to an element of the family $\{U_{\text{hyp}}(\cdot; \xi)\}$

(see Section 5). Hence, at the level $\varepsilon = 0$, there are infinitely many stationary solutions, generating a “finite-time” attracting manifold for the dynamics. Note that, at the hyperbolic level, there is no way of distinguishing one element from any other in the family of steady states $\{U_{\text{hyp}}(\cdot; \xi)\}$ in term of stability properties.

For $\varepsilon > 0$, the situation is different. Apart from the well-known smoothing effect, the presence of the Laplace operator at the right hand-side of (1.1), together with the boundary conditions (1.2), has the effect of a drastic reduction of the number of stationary solutions: from infinitely many to a single stationary state (see [14]). Such solution, denoted here by $\bar{U}_{\text{par}}^\varepsilon = \bar{U}_{\text{par}}^\varepsilon(x)$, converges in the limit $\varepsilon \rightarrow 0^+$ to a specific element of the family $\{U_{\text{hyp}}(\cdot; \xi)\}$. As an example, in the case of the Burgers equation, $f(s) = \frac{1}{2}s^2$, there holds

$$\bar{U}_{\text{par}}^\varepsilon(x) = -\kappa \tanh\left(\frac{\kappa x}{2\varepsilon}\right)$$

where $\kappa = \kappa_\varepsilon(\varepsilon/\ell, u_-)$ is implicitly defined by the relation

$$\kappa(\theta, u) \tanh(\kappa(\theta, u)/2\theta) = u.$$

In the limit $\varepsilon \rightarrow 0^+$, the single steady state $\bar{U}_{\text{par}}^\varepsilon$ converges pointwise to $\bar{U}_{\text{hyp}} := U_{\text{hyp}}(\cdot; 0)$. Therefore, the element of the one-parameter family $\{U_{\text{hyp}}(\cdot; \xi)\}$ corresponding to $\xi = 0$ exhibits a form of *structural stability* which is not shared with any other element of the same family.

A deeper understanding of the problem can be gained by analyzing the dynamical properties of (1.1) for initial data close to the equilibrium configuration by means of the linearized equation at the state $\bar{U}_{\text{par}}^\varepsilon$

$$\partial_t u = \mathcal{L}_\varepsilon u := \varepsilon \partial_x^2 u + \partial_x(a(x)u) \quad \text{with } a(x) := -f'(\bar{U}_{\text{par}}^\varepsilon(x)).$$

In [14] it is shown that, in the case of Burgers flux $f(s) = \frac{1}{2}s^2$, the eigenvalues of \mathcal{L}_ε , considered with homogeneous Dirichlet boundary conditions, are real and negative. Moreover, for $f(u_+) = f(u_-)$, there holds as $\varepsilon \rightarrow 0$

$$\lambda_1^\varepsilon = O(e^{-1/\varepsilon}) \quad \text{and} \quad \lambda_k^\varepsilon < -\frac{c_0}{\varepsilon} < 0 \quad \forall k \geq 2$$

for some $c_0 > 0$ independent on ε . Negativity of the eigenvalues implies that the steady state $\bar{U}_{\text{par}}^\varepsilon$ is asymptotically stable with exponential rate; while the precise distribution shows that the large time behavior is described by term of the order $e^{\lambda_1^\varepsilon t}$ and, as a consequence, the convergence is very slow, when ε is small.

Such is the picture relative to the behavior determined by an initial data close to the equilibrium solution $\bar{U}_{\text{par}}^\varepsilon$. The next question concerns with the dynamics generated by initial data still presenting a sharp transition from u^- to u^+ localized far from the position of the steady state $\bar{U}_{\text{par}}^\varepsilon$. Figure 1 represents the solution to the initial-value problem for (1.1) with boundary conditions (1.2). Starting with a decreasing initial datum, a shock layer is formed in an $\mathcal{O}(1)$ -timescale, so that

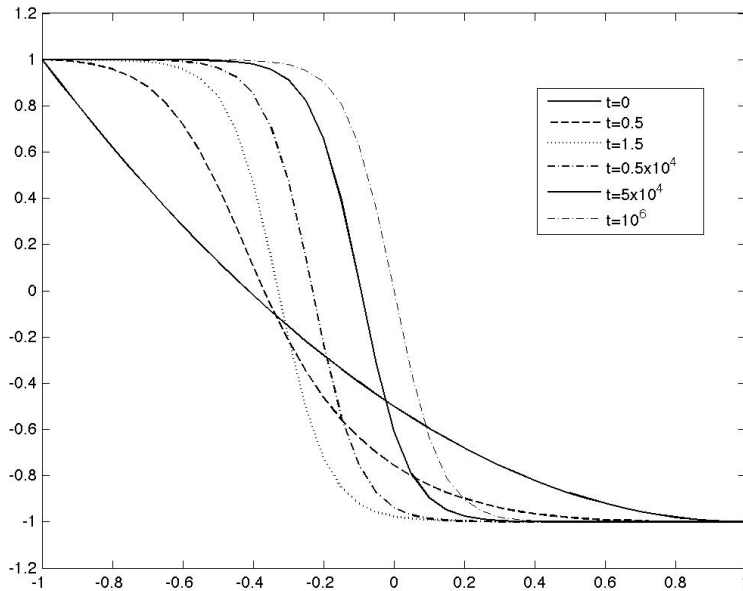


FIGURE 1. The solution to viscous Burgers equation (1.1) with $\varepsilon = 0.04$, $u_{\pm} = \mp 1$ and initial datum $u(x, 0) = u_0(x) := 1/2x^2 - x - 1/2$.

the solution is approximately given by a translation of the (unique) stationary solution of the problem. Once such a layer is formed, it moves towards the location corresponding to the equilibrium solution and this motion is exponentially slow.

Heuristically, since, for $\varepsilon = 0$, there exist infinitely many steady states with a single jump located at an arbitrary point in the interval $(-\ell, \ell)$, for $\varepsilon > 0$ the corresponding profiles—even not being stationary solution—still solves approximately the evolution equation. As a consequence, two different time scales emerge: for short times, the solution becomes closer to a monotone transition connecting the boundary data, close to a space-translation of the single steady state; for long time, the profile slowly moves toward the equilibrium configuration.

To describe the dynamics generated by such an initial configuration and to determine a detailed description of the relation between the unviscous and the low-viscosity behavior, it is rational:

- to build up a one-parameter family of functions $\{U^\varepsilon(\cdot; \xi)\}$ such that $U^\varepsilon(\cdot; \bar{\xi}) = \bar{U}_{\text{par}}^\varepsilon$ for some $\bar{\xi}$, and with the additional property that $U^\varepsilon(\cdot; \xi) \rightarrow U_{\text{hyp}}^\varepsilon(\cdot; \xi)$ as $\varepsilon \rightarrow 0$, in an appropriate sense;
- to describe the dynamics of the viscous scalar conservation law in a tubular neighborhood of the family $\{U^\varepsilon(\cdot; \xi)\}$.

Among others, such problem has been examined in [27] and in [16], where different approaches have been considered. The former is based either on *projection method*

or on *WKB expansions*; the latter stands on an adapted version of the *method of matched asymptotics expansion*. The common aim is to determine an expression and/or an equation for the parameter ξ , considered as a function of time, describing the movement of the transition from a generic point of the interval $(-\ell, \ell)$ toward the equilibrium location $\bar{\xi}$. In both the contributions, the analysis is conducted at a formal level and validated numerically by means of comparison with significant computations. A rigorous analysis has been performed in [7] (and generalized to the case of nonconvex flux in [8]), where one-parameter family of reference functions is chosen as a family of traveling wave solution to the viscous equation satisfying the boundary conditions and with non-zero (but small) velocity. The approach is based on the use of such traveling wave to obtain upper and lower estimates by the maximum principle, from which rigorous asymptotic formulae for the slow velocity are obtained.

Slow motion for the viscous Burgers equation in unbounded domains has been also considered in literature. In [28], it is analyzed the case of the half-line $(0, +\infty)$ for the space variable x , with constant initial and boundary data chosen so that speed of the shock generated at $x = 0$ would be stationary for the hyperbolic equation. The presence of the viscosity generates a motion of the transition layer, which is precisely identified by means of the Lambert's W function. Later, the (slow) motion of a shock wave, with zero hyperbolic speed, for the Burgers equation in the quarter plane has been considered in [19], where it is shown that the location of the wave front is of order $\ln(1+t)$; the same result has been generalized in [23] in the case of general fluxes (for other contributions to the same problem, we refer also to [17, 30]).

The case of the whole real line has been examined in [13] with emphasis on the generation of N -wave like structures and their evolution towards nonlinear diffusion waves. The analysis is based on the use of self-similar variables, suggested by the invariance of the Burgers equation under the group of scaling transformations $(x, t, u) \mapsto (cx, c^2 t, u/c)$ (for subsequent contributions in the same direction, see [12]). More recently, it has been shown in [4] that the slow motion for the viscous Burgers equation on the whole real line is determined by the presence of a one-dimensional center manifold of steady states for the equation in the self-similar variable (corresponding to the diffusion waves) and a relative family of one-dimensional global attractive invariant manifold. In a short-time scale, the solution approaches one of the attractive manifolds and remains close to it in a long-time scale.

At the present day, results relative to metastability in the case of systems appear to be rare. Slow dynamics analysis for systems of conservation laws have been considered in [11], basic model examples being the Navier-Stokes equations of compressible viscous heat conductive fluid and the Keyfitz-Kranzer system, arising in elasticity. The approach is based on asymptotic expansions and consists in deriving appropriate limiting equations for the leading order terms, in the case of periodic data. In [15], the problem of proving convergence to a stationary solution for a system of conservation laws with viscosity is addressed, with an approach based on

a detailed analysis of the linearized operator at the steady state. A recent contribution is the reference [3], where the authors consider the Saint-Venant equations for shallow water and, precisely, the phenomenon of formation of roll-waves. The approach merges together analytical techniques and numerical results to present some intriguing properties relative to the dynamics of solitary wave pulses.

Summing up, apart for the formal expansions methods, the rigorous approaches used in the literature are largely based on typical scalar equations features. The first of these properties is the direct link between the scalar Burgers equation and the heat equation given by the Hopf–Cole transformation: $u = -2\varepsilon \phi^{-1} \partial_x \phi$, and the consequent invariance of equation (1.1) under the group of scaling transformations $(x, t, u) \mapsto (cx, c^2 t, u/c)$. On the one hand, the presence of such a connection is an evident advantage, since it permits to determine optimal descriptions for the behavior under study (see [13, 19, 28]); on the other hand, to use such exceptional property makes the approach very stiff and difficult to apply to more general cases. A different “scalar hallmark” is the base of the approach considered in [7], where the authors make wide use of maximum principle and comparison arguments, taking benefit from the fact that the equation is second-order parabolic.

In order to extend the results to more general settings and specifically for systems of PDEs, it is useful to determine strategies and techniques that are more flexible, paying, if necessary, the price of a less accurate description of the dynamics. A contribution in this direction has been given in [23], where the location of the shock transition for a scalar conservation law in the quarter plane has been proved by means of weighted energy estimates, extending the result proved in [19], that used an explicit formula –determined by means of the Hopf–Cole transformation– for the Green function of the linearization at the shock profile of the Burgers equation.

The essence of this article is to contribute to the definition of a flexible language, hopefully relevant to more general contexts and, mainly, in the case of systems. With this direction in mind, we follow an approach that it is strictly related with the *projection method* considered in [5, 27] and we go behind the philosophy tracked in the analysis of stability of viscous shock waves by K.Zumbrun and co-authors (see [31, 22, 21]). Precisely, once a set of reference states $\{U^\varepsilon(\cdot; \xi)\}$ is chosen, we separate two distinct phases:

- to determine spectral properties of the linearized operator at such states;
- to show that appropriate assumptions on the structure of the eigenvalues of such operators together with a control on how far is the approximate state from being an exact solution imply the presence of a metastable behavior.

The main advantage in such a separation stems from the fact that, in principle, it should be possible to obtain numerical evidence of special spectral structures in cases where analytical results appear to be not achievable.

With respect to the framework of shock waves stability analysis, there are two main differences. First of all, we concentrate on the case of bounded domains and, therefore, the spectrum of the linearized operator is discrete and given by a divergent sequence of (real) values. Additionally, the reference states U^ε generically are approximate solutions, in the sense that they satisfy the steady state equation with an error, that converges to zero as $\varepsilon \rightarrow 0^+$. Hence the perturbations of such states satisfy at first order a *non-homogeneous* linear equation, with forcing term, formally negligible as $\varepsilon \rightarrow 0^+$.

Our approach consists in approximating the evolution of the couple (ξ, v) , where ξ denotes the parameter for the approximate manifold and v the perturbation of the profile U^ε , by a partial linearization, giving raise to a system which we call **quasi-linear system**. This is obtained by linearizing with respect to v and keeping the nonlinear dependence on ξ , in order to keep track of the nonlinear evolution along the approximate manifold. The main contribution of the paper is Theorem 3.1, stating that, assuming a number of assumptions relative to the elements of the approximate manifold U^ε and the linearized operator at such states, and a coupling condition, linking the first eigenvalue of the linearized operator with the nonlinear operator evaluated at U^ε , the solution to the quasi-linearized system is described by the evolution of a reduced system where the equation for ξ is decoupled from w and hence solvable by means of the standard separation of variable method. Details on assumptions and statement are given in Section 3.

We close the Introduction with an overview on the structure of the article. In Section 2, we introduce a general framework to describe the dynamics in a cylindrical neighborhood of an approximate manifold $\{U^\varepsilon(\cdot; \xi)\}$ of solutions for scalar evolution equations, using as coordinates the parameter ξ , describing the manifold, and a distance vector v , determined by the difference between the solution u and the element of the manifold. The couple (ξ, v) turns to solve an ODE-PDE coupled system of equations and, in Section 3, we concentrate on the analysis of an approximation of the system, obtained by disregarding $o(v)$ - terms, which we call **quasi-linearized system**, since it still preserves nonlinear dependence on ξ . In Section 4, we analyze spectral properties of the diffusion-transport linear operator, arising from the linearization at the state $U^\varepsilon(\cdot; \xi)$. Specifically, we introduce our basic assumptions needed to state and prove Theorem 3.1. We show that, under appropriate assumption on the limiting behavior of U^ε as $\varepsilon \rightarrow 0^+$, the spectrum can be decomposed into two parts: the first eigenvalue turns of order $O(e^{-C/\varepsilon})$, $C > 0$, hence small as $\varepsilon \rightarrow 0^+$; all of the remaining eigenvalues are less than $-C/\varepsilon$, $C > 0$. Such estimates will translate in the following Sections into a one-dimensional dynamics, since all of the components relative all of the eigenvector except the first will have a very fast decay for ε small, and in a slow motion along the approximate manifold, as a consequence of the size estimate for the first eigenvalue. Finally, in Appendix (Section 5), we prove that in the purely hyperbolic case, i.e. $\varepsilon = 0$, the entropic

solution to the initial-boundary value problem converge in finite time to one of the jump configurations $U_{\text{hyp}}(\cdot; \xi)$ for some ξ .

2. GENERAL FRAMEWORK

Let us consider the Cauchy problem for a general evolution equation

$$(2.1) \quad \partial_t u = \mathcal{F}^\varepsilon[u], \quad u|_{t=0} = u_0$$

where \mathcal{F}^ε denote a nonlinear differential operator, depending singularly on the parameter $\varepsilon > 0$, so that the formal limiting problem $\partial_t u = \mathcal{F}^0[u]$ is of lower order. Typically, equation (2.1) is complemented with appropriate boundary conditions, appearing in the definition of an appropriate Hilbert space X such that a solution to (2.1) is a function $u : [0, +\infty) \rightarrow X$.

Denoting by $u^\varepsilon = u^\varepsilon(x, t)$ the solution of (2.1), we are interested in describing the behavior of u^ε for small ε .

The basic example we have in mind is the initial-boundary value problem for a scalar conservation law with viscosity with Dirichlet boundary condition in the bounded interval $I = (-\ell, \ell)$, that is

$$(2.2) \quad \begin{cases} \partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u & x \in I, t \geq 0 \\ u(\pm \ell, t) = u_\pm & t \geq 0 \\ u(x, 0) = u_0(x) & x \in I \end{cases}$$

for some $\varepsilon, \ell > 0$, $u_\pm \in \mathbb{R}$ and flux function f , chosen so that assumptions (1.5) hold.

We assume to have a one-parameter family $\{U^\varepsilon(\cdot; \xi)\}$ in X , parametrized by $\xi \in I$, whose elements are approximate stationary solution to the problem, i.e. $\mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] \rightarrow 0$ as $\varepsilon \rightarrow 0$. Precisely, we assume that the term $\mathcal{F}^\varepsilon[U^\varepsilon]$ belongs to the dual space of the continuous functions space $C(I)$ and there exists a family of smooth positive functions $\Omega^\varepsilon = \Omega^\varepsilon(\xi)$, uniformly convergent to zero as $\varepsilon \rightarrow 0$, such that, for any $\xi \in I$, there holds

$$|\langle \psi(\cdot), \mathcal{F}^\varepsilon[U^\varepsilon(\cdot, \xi)] \rangle| \leq \Omega^\varepsilon(\xi) \|\psi\|_{L^\infty} \quad \forall \psi \in C(I).$$

The family $\{U^\varepsilon(x; \xi)\}_{\xi \in I}$ will be referred to as an **approximate invariant manifold** with respect to the flow determined by (2.1) in the Hilbert space X .

The dependence of Ω^ε on ε plays a fundamental rôle, since it drives the departure from the approximate invariant manifold. In the specific case under consideration, such term is *exponentially small*, meaning that it behaves as $e^{-C/\varepsilon}$, with $C > 0$.

Let us stress that, differently to the construction in [16] and in [27], where the approximate solutions satisfies exactly the equation and the boundary condition to within exponentially small terms, here we assume the elements U^ε to satisfy the boundary conditions exactly and the equation approximately.

Example 2.1. In the case of Burgers equation, i.e. $f(s) = \frac{1}{2} s^2$ and $u^\pm := \mp u_*$, for some $u_* > 0$, we consider a function obtained by matching two different steady

states satisfying, respectively, the left and the right boundary condition together with the request $U(\xi) = 0$; in formulas,

$$U^\varepsilon(x; \xi) = \begin{cases} \kappa_- \tanh(\kappa_- (\xi - x)/2\varepsilon) & \text{in } (-\ell, \xi) \\ \kappa_+ \tanh(\kappa_+ (\xi - x)/2\varepsilon) & \text{in } (\xi, \ell), \end{cases}$$

where κ_\pm are chosen so that the boundary conditions are satisfied

$$(2.3) \quad \kappa_\pm \tanh\left(\frac{\kappa_\pm}{2\varepsilon}(\xi \mp \ell)\right) = u_\pm.$$

By direct substitution, denoting by $\delta_{x=\xi}$ the usual Dirac's delta distribution concentrated at $x = \xi$, we obtain the identity

$$\mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] = \llbracket \partial_x U^\varepsilon \rrbracket_{x=\xi} \delta_{x=\xi}$$

in the sense of distributions. In particular, $U^\varepsilon(\cdot, \xi)$ is a stationary solution if and only if $\xi = 0$, that corresponds to the equilibrium position for the shock layer location in the case of the Burgers flux. Going further, by differentiation, we have

$$\llbracket \partial_x U^\varepsilon \rrbracket_{x=\xi} = \frac{1}{2\varepsilon}(\kappa_- - \kappa_+)(\kappa_- + \kappa_+).$$

In order to determine the behavior of Ω^ε for small ε , we need an asymptotic description of the values κ_\pm . To this aim, let us set $\kappa_\pm := \mp u_\pm(1 + h_\pm)$, so that, denoting by $\Delta_\pm := \ell \mp \xi$ the distance from ξ to $\pm\ell$, relation (2.3) becomes

$$\tanh\left(\mp \frac{u_\pm \Delta_\pm}{2\varepsilon}(1 + h_\pm)\right) = \frac{1}{1 + h_\pm}.$$

Therefore, the values h_\pm are both positive and thus

$$\tanh\left(\mp \frac{u_\pm \Delta_\pm}{2\varepsilon}\right) \leq \frac{1}{1 + h_\pm}.$$

that gives the asymptotic representation

$$h_\pm \leq \frac{1}{\tanh(\mp u_\pm \Delta_\pm / 2\varepsilon)} - 1 = \frac{2}{e^{\mp u_\pm \Delta_\pm / \varepsilon} - 1} = 2e^{\pm u_\pm \Delta_\pm / \varepsilon} + l.o.t.,$$

where *l.o.t.* denotes lower order terms. Thus, we infer

$$\begin{aligned} \llbracket \partial_x U^\varepsilon \rrbracket_{x=\xi} &= \frac{u_*^2}{2\varepsilon}(h_- - h_+)(2 + h_- + h_+) = \frac{u_*^2}{\varepsilon}(h_- - h_+) + l.o.t. \\ &= \frac{2u_*^2}{\varepsilon}(e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon}) + l.o.t. \sim C\xi e^{-C/\varepsilon}, \end{aligned}$$

showing that the term $\llbracket \partial_x U^\varepsilon \rrbracket_{x=\xi}$ is null at $\xi = 0$ and exponentially small for $\varepsilon \rightarrow 0^+$.

Example 2.2. A different example of approximated manifold for a Burgers type equation can be obtained by following the traveling wave approach considered in [7]. In this case, we look for a solution to (2.2) of the form

$$(2.4) \quad u(x, t) = \Phi^\varepsilon(x - V^\varepsilon(\xi)t), \quad \text{satisfying} \quad \Phi^\varepsilon(\pm\ell) = \mp u_*, \quad \Phi^\varepsilon(\xi) = 0$$

for some $\xi \in (-\ell, \ell)$ and for some $u_* > 0$. Such solution moves with speed V and satisfies the boundary condition only if $t = 0$. Then the function Φ^ε has to satisfy

$$(2.5) \quad \varepsilon \partial_x^2 \Phi^\varepsilon + (\Phi^\varepsilon + V) \partial_x \Phi^\varepsilon = 0, \quad \Phi^\varepsilon(\pm \ell) = \mp u_*$$

Since $\mathcal{F}[\Phi^\varepsilon] = \partial_t \Phi^\varepsilon$, by (2.4) we get

$$\mathcal{F}[\Phi^\varepsilon(\cdot; \xi)] = -V^\varepsilon(\xi) \cdot \partial_x \Phi^\varepsilon(\cdot; \xi).$$

In order to determine the behavior of $\mathcal{F}[\Phi^\varepsilon]$ we need an asymptotic description of the speed V in terms of ε . As proved in [7], for any $\xi \in (-\ell, \ell)$ and any $\varepsilon > 0$ there exists a unique solution Φ^ε of (2.5), satisfying for $\varepsilon \rightarrow 0$ the asymptotic

$$\begin{aligned} \Phi^\varepsilon(x) &= \mp u_* + O(\exp(-u_*|\xi - x|/\varepsilon) + R_\varepsilon) \quad \text{for } x \neq \xi \\ V^\varepsilon(\xi) &= u_*[\exp(-u^*(\ell - \xi)/\varepsilon) + \exp(-u_*(\ell + \xi)/\varepsilon)] + O(R_\varepsilon^2/\varepsilon) \end{aligned}$$

where R_ε is defined by

$$R_\varepsilon := \exp(-u_*(\ell - \xi)/\varepsilon) + \exp(-u_*(\ell + \xi)/\varepsilon)$$

Thus we have an asymptotic representation for the speed V

$$V^\varepsilon \sim u_* e^{-u_*(\ell + \xi)/\varepsilon} - u_* e^{-u^*(\ell - \xi)/\varepsilon} + l.o.t.$$

which coincides, up to a multiplicative factor, with the formula determined in Example 2.1.

Once the one-parameter family $\{U^\varepsilon(\cdot; \xi)\}$ is chosen, we write the solution to the initial value problem (2.1) as

$$u(\cdot, t) = U^\varepsilon(\cdot; \xi(t)) + v(\cdot, t)$$

with $\xi = \xi(t) \in I$ and $v = v(\cdot, t) \in X$ to be determined. Substituting, we obtain

$$(2.6) \quad \partial_t v = \mathcal{L}_\xi^\varepsilon v + \mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] - \partial_\xi U^\varepsilon(\cdot; \xi) \frac{d\xi}{dt} + \mathcal{Q}^\varepsilon[v, \xi]$$

where

$$\begin{aligned} \mathcal{L}_\xi^\varepsilon v &:= d\mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] v \\ \mathcal{Q}^\varepsilon[v, \xi] &:= \mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi) + v] - \mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] - d\mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] v. \end{aligned}$$

Next, let us assume that, for any ξ , the linear operator $\mathcal{L}_\xi^\varepsilon$ has an increasing sequence of real eigenvalues $\lambda_k^\varepsilon = \lambda_k^\varepsilon(\xi)$ with $\lambda_k \rightarrow -\infty$ as $k \rightarrow +\infty$ with corresponding right eigenfunctions $\phi_k^\varepsilon = \phi_k^\varepsilon(\cdot; \xi)$. Denoting by $\psi_k^\varepsilon = \psi_k^\varepsilon(\cdot; \xi)$ the eigenfunctions of the corresponding adjoint operator $\mathcal{L}_\xi^{\varepsilon,*}$ and setting

$$v_k = v_k(\xi; t) := \langle \psi_k^\varepsilon(\cdot; \xi), v(\cdot, t) \rangle,$$

we can use the degree of freedom we still have in the choice of the couple (v, ξ) in such a way that component v_1 is identically zero, that is

$$\frac{d}{dt} \langle \psi_1^\varepsilon(\cdot; \xi(t)), v(\cdot, t) \rangle = 0 \quad \text{and} \quad \langle \psi_1^\varepsilon(\cdot; \xi_0), v_0(\cdot) \rangle = 0.$$

Using equation (2.6), we infer

$$\langle \psi_1^\varepsilon(\xi, \cdot), \mathcal{L}_\xi^\varepsilon v + \mathcal{F}[U^\varepsilon(\cdot; \xi)] - \partial_\xi U^\varepsilon(\cdot; \xi) \frac{d\xi}{dt} + \mathcal{Q}^\varepsilon[v, \xi] \rangle + \langle \partial_\xi \psi_1^\varepsilon(\xi, \cdot) \frac{d\xi}{dt}, v \rangle = 0$$

Since $\langle \psi_1^\varepsilon, \mathcal{L}_\xi^\varepsilon v \rangle = \lambda_1 \langle \psi_1^\varepsilon, v \rangle$, we obtain a scalar differential equation for the variable ξ , describing the reduced dynamics along the approximate manifold, that is

$$(2.7) \quad \alpha^\varepsilon(\xi, v) \frac{d\xi}{dt} = \langle \psi_1^\varepsilon(\cdot; \xi), \mathcal{F}[U^\varepsilon(\cdot; \xi)] + \mathcal{Q}^\varepsilon[v, \xi] \rangle$$

where

$$\alpha_0^\varepsilon(\xi) := \langle \psi_1^\varepsilon(\cdot; \xi), \partial_\xi U^\varepsilon(\cdot; \xi) \rangle \quad \text{and} \quad \alpha^\varepsilon(\xi, v) := \alpha_0^\varepsilon(\xi) - \langle \partial_\xi \psi_1^\varepsilon(\cdot; \xi), v \rangle,$$

together with the condition on the initial datum ξ_0

$$\langle \psi_1^\varepsilon(\cdot; \xi_0), v_0(\cdot) \rangle = 0$$

To rewrite equation (2.7) in normal form in the regime of small v , we assume

$$|\alpha_0^\varepsilon(\xi)| = |\langle \psi_1^\varepsilon(\cdot; \xi), \partial_\xi U^\varepsilon(\cdot; \xi) \rangle| \geq c_0 > 0 \quad \forall \xi \in I.$$

for some $c_0 > 0$. Such assumption gives a (weak) restriction on the choice of the members of the family $\{U^\varepsilon\}$ asking for the manifold to be never transversal to the first eigenfunction of the corresponding linearized operator. From now on, thanks to the previous hypothesis, we can renormalize the eigefunction ψ_1^ε so that

$$\alpha_0^\varepsilon(\xi) = \langle \psi_1^\varepsilon(\cdot; \xi), \partial_\xi U^\varepsilon(\cdot; \xi) \rangle = 1 \quad \forall \varepsilon > 0, \xi \in I.$$

Since we are interested in the regime $v \rightarrow 0$, we expand $1/\alpha^\varepsilon$ as

$$\frac{1}{\alpha^\varepsilon(\xi, v)} = \frac{1}{\alpha_0^\varepsilon(\xi)} \left(1 + \frac{\langle \partial_\xi \psi_1^\varepsilon, v \rangle}{\alpha_0^\varepsilon(\xi)} \right) + o(|v|) = 1 + \langle \partial_\xi \psi_1^\varepsilon, v \rangle + o(|v|).$$

Inserting in (2.7), we end up with the nonlinear equation for ξ

$$(2.8) \quad \frac{d\xi}{dt} = \theta^\varepsilon(\xi) (1 + \langle \partial_\xi \psi_1^\varepsilon, v \rangle) + \rho^\varepsilon[\xi, v],$$

where

$$(2.9) \quad \theta^\varepsilon(\xi) := \langle \psi_1^\varepsilon, \mathcal{F}[U^\varepsilon] \rangle$$

$$(2.10) \quad \rho^\varepsilon[\xi, v] := \frac{1}{\alpha^\varepsilon(\xi, v)} (\langle \psi_1^\varepsilon, \mathcal{Q}^\varepsilon \rangle + \langle \partial_\xi \psi_1^\varepsilon, v \rangle^2).$$

Using (2.8), the equation (2.6) for v can be rephrased as

$$(2.11) \quad \partial_t v = H^\varepsilon(x; \xi) + (\mathcal{L}_\xi^\varepsilon + \mathcal{M}_\xi^\varepsilon) v + \mathcal{R}^\varepsilon[v, \xi]$$

where

$$\begin{aligned} H^\varepsilon(x; \xi) &:= \mathcal{F}^\varepsilon[U^\varepsilon(x; \xi)] - \partial_\xi U^\varepsilon(x; \xi) \theta^\varepsilon(\xi), \\ \mathcal{M}_\xi^\varepsilon v &:= -\partial_\xi U^\varepsilon(\cdot; \xi) \theta^\varepsilon(\xi) \langle \partial_\xi \psi_1^\varepsilon, v \rangle, \\ \mathcal{R}^\varepsilon[v, \xi] &:= \mathcal{Q}^\varepsilon[v, \xi] - \partial_\xi U^\varepsilon(\cdot; \xi) \rho^\varepsilon[\xi, v]. \end{aligned}$$

Let us stress that, by definition, there holds

$$(2.12) \quad \langle \psi_1^\varepsilon(\cdot; \xi), H^\varepsilon(\cdot; \xi) \rangle = 0,$$

so that $H^\varepsilon(\cdot; \xi)$ is the projection of $\mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)]$ onto the space orthogonal to $\phi_1^\varepsilon(\cdot; \xi)$.

To show how such formulas can be handled, at least formally, in concrete situations, let us analyze problem (2.2), namely

$$\mathcal{F}^\varepsilon[u] = \varepsilon \partial_x^2 u - \partial_x f(u)$$

for $\varepsilon > 0$ and f satisfying assumptions (1.5). Retracing the definitions introduced above and setting $a^\varepsilon(x; \xi) := f'(U^\varepsilon(x; \xi))$, we get the following expressions

$$\mathcal{L}_\xi^\varepsilon v := \varepsilon \partial_x^2 v - \partial_x (a^\varepsilon(\cdot; \xi) v) \quad \mathcal{L}_\xi^{\varepsilon,*} v := \varepsilon \partial_x^2 v + a^\varepsilon(\cdot; \xi) \partial_x v$$

where the adjoint operator $\mathcal{L}_\xi^{\varepsilon,*}$ has to be considered with Dirichlet boundary conditions, and

$$\mathcal{Q}^\varepsilon[v, \xi] := -\partial_x \mathcal{N}^\varepsilon[v, \xi] = -\partial_x \left\{ f(U^\varepsilon(\cdot; \xi) + v) - f(U^\varepsilon(\cdot; \xi)) - f'(U^\varepsilon(\cdot; \xi)) v \right\}$$

where $\mathcal{N}^\varepsilon = o(|v|)$, so that $\mathcal{Q}^\varepsilon = o(|v|, |\partial_x v|)$. Formally, for small ε and small v , the dynamics of the parameter ξ is approximately given by

$$\frac{d\xi}{dt} = \theta^\varepsilon(\xi) + \dots,$$

with θ^ε given in (2.9). Next, we need to identify the functions ψ_1^ε and $\partial_\xi U^\varepsilon$ in the limiting regime $\varepsilon \rightarrow 0$, at least approximately. For $\varepsilon \sim 0$, the function ψ_1^ε is close to the eigenfunction of the operator $\mathcal{L}_\xi^{0,*}$ relative to the eigenvalue $\lambda = 0$, with

$$a^0(x; \xi) := u_- \chi_{(-\ell, \xi)}(x) + u_+ \chi_{(\xi, \ell)}(x)$$

Hence, we obtain the representation formula

$$\psi_1^\varepsilon(x) \sim \psi_1^0(x) := \begin{cases} (1 - e^{u_+(\ell-\xi)/\varepsilon})(1 - e^{-u_-(\ell+x)/\varepsilon}) & x < \xi, \\ (1 - e^{-u_-(\ell+\xi)/\varepsilon})(1 - e^{u_+(\ell-x)/\varepsilon}) & x > \xi, \end{cases}$$

so that $\psi_1^\varepsilon \sim 1$, provided ξ is bounded away from the boundaries $\pm\ell$. Additionally, with the approximation $U^\varepsilon(x; \xi) \sim U_{\text{hyp}}(x; \xi)$, defined in (1.6), we infer

$$\frac{U^\varepsilon(x; \xi + h) - U^\varepsilon(x; \xi)}{h} \sim -\frac{1}{h} \llbracket u \rrbracket \chi_{(\xi, \xi+h)}(x)$$

so that we expect $\partial_\xi U^\varepsilon$ to converge to $-\llbracket u \rrbracket \delta_\xi$ as $\varepsilon \rightarrow 0$ in the sense of distributions, so that $\alpha_0^\varepsilon(\xi) \sim -\llbracket u \rrbracket$. Therefore, we deduce an (approximate) expression for the function θ^ε

$$\theta^\varepsilon(\xi) \sim -\frac{1}{\llbracket u \rrbracket} \langle 1, \mathcal{F}[U^\varepsilon] \rangle,$$

that, with the choice of U^ε proposed in Example 2.1, reduces to

$$(2.13) \quad \theta^\varepsilon(\xi) \sim \frac{1}{\varepsilon} u_* (e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon}),$$

which coincides with the corresponding formula determined in [27].

3. ANALYSIS OF THE QUASI-LINEARIZED SYSTEM

Next, let us go back to the system (2.8)–(2.11) for the couple (ξ, v) and let us neglect the $o(v)$ order terms:

$$(3.1) \quad \begin{cases} \frac{d\zeta}{dt} = \theta^\varepsilon(\zeta)(1 + \langle \partial_\xi \psi_1^\varepsilon, w \rangle), \\ \partial_t w = H^\varepsilon(\zeta) + (\mathcal{L}_\xi^\varepsilon + \mathcal{M}_\xi^\varepsilon)w \end{cases}$$

to be complemented with initial conditions

$$(3.2) \quad \zeta(0) = \zeta_0 \in (-\ell, \ell) \quad \text{and} \quad w(x, 0) = w_0(x) \in L^2(I).$$

From now on, we will refer to this system as the **quasi-linearization** of (2.8)–(2.11). We are interested in describing the behavior of the solution to such system in the regime of small ε .

Shortly, the quasi-linearized system is determined by an appropriate combination of the term $\mathcal{F}^\varepsilon[U^\varepsilon]$, measuring how far is the function U^ε from being a stationary solution, and the linear operator $\mathcal{L}_\xi^\varepsilon$, controlling at first order how solutions to (2.1) depart from U^ε when the latter is taken as initial datum. To state our first result, we need to precise the assumption on such terms.

H1. The family $\{U^\varepsilon(\cdot, \xi)\}$ is such that $\mathcal{F}^\varepsilon[U^\varepsilon]$ belongs to the dual space of $C(I)$ and there exists a family of smooth positive functions Ω^ε such that

$$|\langle \psi(\cdot), \mathcal{F}^\varepsilon[U^\varepsilon(\cdot, \xi)] \rangle| \leq \Omega^\varepsilon(\xi) \|\psi\|_{L^\infty} \quad \forall \psi \in C(I).$$

with Ω^ε converging to zero as $\varepsilon \rightarrow 0$, uniformly with respect to ξ .

H2. Let $\{\dots < \lambda_k^\varepsilon(\xi) < \dots < \lambda_2^\varepsilon(\xi) < \lambda_1^\varepsilon(\xi)\}$ be the sequence of eigenvalues of the linear operator $\mathcal{L}_\xi^\varepsilon$. Assume that for any $\xi \in (-\ell, \ell)$ there hold

$$\lambda_1^\varepsilon(\xi) - \lambda_2^\varepsilon(\xi) \geq C, \quad \lambda_1^\varepsilon(\xi) < 0 \quad \lambda_k^\varepsilon(\xi) \leq -C k^2 \quad \text{with } k \geq 2.$$

for some constant $C > 0$ independent on k , ε and ξ .

H3. Given $\xi \in I$, let $\phi_k^\varepsilon(\cdot; \xi)$ and $\psi_k^\varepsilon(\cdot; \xi)$ be a sequence of eigenfunction for the operators $\mathcal{L}_\xi^\varepsilon$ and $\mathcal{L}_\xi^{\varepsilon,*}$, respectively, normalized so that

$$(3.3) \quad \langle \psi_1^\varepsilon(\cdot; \xi), \partial_\xi U^\varepsilon(\cdot; \xi) \rangle = 1 \quad \text{and} \quad \langle \psi_j^\varepsilon, \phi_k^\varepsilon \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

Then we assume

$$(3.4) \quad \sum_j \langle \partial_\xi \psi_k^\varepsilon, \phi_j^\varepsilon \rangle^2 = \sum_j \langle \psi_k^\varepsilon, \partial_\xi \phi_j^\varepsilon \rangle^2 \leq C \quad \forall k.$$

for some constant C independent on the parameter ξ .

For later use, note that, by differentiation, there holds

$$(3.5) \quad \langle \partial_\xi \psi_j^\varepsilon, \phi_k^\varepsilon \rangle + \langle \psi_j^\varepsilon, \partial_\xi \phi_k^\varepsilon \rangle = 0.$$

Also, we use the notation $\Lambda_k^\varepsilon := \sup_{\xi \in I} \lambda_k^\varepsilon(\xi)$.

Theorem 3.1. *Let hypotheses **H1-2-3** be satisfied. Additionally, assume that*

$$(3.6) \quad \Omega^\varepsilon(\xi) \leq C |\lambda_1^\varepsilon(\xi)|, \quad \forall \xi \in (-\ell, \ell)$$

for some constant $C > 0$ independent on ε and ξ .

Then, denoted by (ζ, w) the solution to the initial-value problem (3.1)–(3.2), for any ε sufficiently small, there exists a time T^ε such that for any $t \leq T^\varepsilon$ the solution w can be represented as

$$w = z + R$$

where z is defined by

$$z(x, t) := \sum_{k \geq 2} w_k(0) \exp \left(\int_0^t \lambda_k^\varepsilon(\zeta(\sigma)) d\sigma \right) \phi_k^\varepsilon(x; \zeta(t)),$$

and the remainder R satisfies the estimate

$$(3.7) \quad |R|_{L^2} \leq C |\Omega^\varepsilon|_{L^\infty} \left\{ \exp \left(\int_0^t \lambda_1^\varepsilon(\zeta(\sigma)) d\sigma \right) |w_0|_{L^2} + 1 \right\}$$

for some constant $C > 0$.

Moreover, for w_0 sufficiently small in L^2 , the final time T^ε can be chosen of the order $-C |\Omega^\varepsilon|_{L^\infty}^{-1} \ln |\Omega^\varepsilon|_{L^\infty}$.

The conclusion of the proof of Theorem 3.1 is based on the following version of a standard nonlinear iteration argument.

Lemma 3.2. *Let $f = f(t)$, $g = g(t)$ and $h = h(s, t)$ be continuous functions for $t \in [0, T]$ for some $T > 0$, such that*

$$f(t) \geq 0, \quad g(t) > 0, \quad g \text{ decreasing}, \quad h(s, t) \geq 0.$$

Let $y = y(t)$ be a non-negative function satisfying the estimate

$$y(t) \leq \int_0^t \{f(s) g(t) y^2(s) + h(s, t)\} ds$$

for any $t \leq T$. Then, if, for any $t \in [0, T]$ there holds

$$(3.8) \quad \sup_{t \in [0, T]} \int_0^t g^2(s) f(s) ds \cdot \sup_{t \in [0, T]} g^{-1}(t) \int_0^t h(s, t) ds < \frac{1}{4}$$

then

$$y(t) \leq 2 \sup_{\tau \in [0, t]} \int_0^\tau h(s, \tau) ds$$

for any $t \in [0, T]$.

Proof of Lemma 3.2. The auxiliary function $w(t) := g^{-1}(t) y(t)$ enjoys the estimate

$$w(t) \leq \int_0^t \{\alpha(s) w^2(s) + \beta(s, t)\} ds$$

where $\alpha(t) := f(t) g^2(t)$ and $\beta(s, t) = g^{-1}(t) h(s, t)$. Set

$$N(t) := \sup_{\tau \in [0, t]} w(\tau).$$

Then, for any $t \in [0, T]$, there holds

$$w(t) \leq \left(\int_0^t \alpha(s) ds \right) N^2(T) + \int_0^t \beta(s, t) ds$$

and, as a consequence, also

$$N(T) \leq A N^2(T) + B$$

where

$$A = A(T) := \sup_{t \in [0, T]} \int_0^t \alpha(s) ds, \quad B = B(T) := \sup_{t \in [0, T]} \int_0^t \beta(s, t) ds.$$

Since $N(0) = 0$, if $1 - 4AB > 0$, then

$$N < \frac{1 - \sqrt{1 - 4AB}}{2A} = \frac{2B}{1 + \sqrt{1 - 4AB}} \leq 2B.$$

In term of y , if (3.8) holds, then

$$y(t) < 2g(t) \sup_{t \in [0, T]} g^{-1}(t) \int_0^t h(s, t) ds.$$

The final estimate follows from the monotonicity of the function g . □

Lemma 3.2 gives the final step needed to prove the Theorem.

Proof of Theorem 3.1. Setting

$$w(x, t) = \sum_j w_j(t) \phi_j^\varepsilon(x, \zeta(t)),$$

we obtain an infinite-dimensional differential system for the coefficients w_j

$$(3.9) \quad \frac{dw_k}{dt} = \lambda_k^\varepsilon(\zeta) w_k + \langle \psi_k^\varepsilon, F \rangle$$

where, omitting the dependencies for shortness,

$$F := H^\varepsilon + \sum_j w_j \left\{ \mathcal{M}_\zeta^\varepsilon \phi_j^\varepsilon - \partial_\xi \phi_j^\varepsilon \frac{d\zeta}{dt} \right\} = H^\varepsilon - \theta^\varepsilon \sum_j \left(a_j + \sum_\ell b_{j\ell} w_\ell \right) w_j.$$

and the coefficients a_j , b_{jk} are given by

$$a_j := \langle \partial_\xi \psi_1^\varepsilon, \phi_j^\varepsilon \rangle \partial_\xi U^\varepsilon + \partial_\xi \phi_j^\varepsilon, \quad b_{j\ell} := \langle \partial_\xi \psi_1^\varepsilon, \phi_\ell^\varepsilon \rangle \partial_\xi \phi_j^\varepsilon$$

Convergence of the series is guaranteed by assumption (3.4).

By (3.5), for the coefficients a_j there hold

$$\langle \psi_k^\varepsilon, a_j \rangle = \langle \partial_\xi \psi_1^\varepsilon, \phi_j^\varepsilon \rangle (\langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle - 1),$$

so that we can also take advantage from the relation $\langle \psi_1^\varepsilon, a_j \rangle = 0$ for any j . Thanks to these relations, equation (3.9) for $k = 1$ simplifies to

$$(3.10) \quad \frac{dw_1}{dt} = \lambda_1^\varepsilon(\zeta) w_1 - \theta^\varepsilon(\zeta) \sum_{\ell, j} \langle \psi_1^\varepsilon, b_{j\ell} \rangle w_\ell w_j$$

Now let us set

$$E_k(s, t) := \exp \left(\int_s^t \lambda_k^\varepsilon(\zeta(\sigma)) d\sigma \right).$$

Note that, for $0 \leq s < t$, there hold

$$E_k(s, t) = \frac{E_k(0, t)}{E_k(0, s)} \quad \text{and} \quad 0 \leq E_k(s, t) \leq e^{\Lambda_k(t-s)}.$$

From equalities (3.10) and (3.9), choosing $w_1(0) = 0$, there follow

$$\begin{aligned} w_1(t) &= - \int_0^t \theta^\varepsilon(\zeta) \sum_{\ell, j} \langle \psi_1^\varepsilon, b_{j\ell} \rangle w_\ell w_j E_1(s, t) ds \\ w_k(t) &= w_k(0) E_k(0, t) \\ &\quad + \int_0^t \left\{ \langle \psi_k^\varepsilon, H^\varepsilon \rangle - \theta^\varepsilon(\zeta) \sum_j \left(\langle \psi_k^\varepsilon, a_j \rangle + \sum_\ell \langle \psi_k^\varepsilon, b_{j\ell} \rangle w_\ell \right) w_j \right\} E_k(s, t) ds, \end{aligned}$$

for $k \geq 2$. Such expressions suggest to introduce the function

$$z(x, t) := \sum_{k \geq 2} w_k(0) E_k(0, t) \phi_k^\varepsilon(x; \zeta(t)),$$

which satisfies the estimate $|z|_{L^2} \leq |w_0|_{L^2} e^{\Lambda_2^\varepsilon t}$.

From the representation formulas for the coefficients w_k , since

$$|\theta^\varepsilon(\zeta)| \leq C \Omega^\varepsilon(\zeta) \quad \text{and} \quad |\langle \psi_k^\varepsilon, H^\varepsilon \rangle| \leq C \Omega^\varepsilon(\zeta) \{1 + |\langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle|\}$$

for some constant $C > 0$ depending on the L^∞ -norm of ψ_k^ε , there holds

$$\begin{aligned} |w - z|_{L^2}^2 &\leq C \left(\int_0^t \Omega^\varepsilon(\zeta) \sum_j |\langle \psi_1^\varepsilon, \partial_\xi \phi_j^\varepsilon \rangle| |w_j| \sum_\ell |\langle \partial_\xi \psi_1^\varepsilon, \phi_\ell^\varepsilon \rangle| |w_\ell| E_1(s, t) ds \right)^2 \\ &\quad + C \sum_{k \geq 2} \left(\int_0^t \Omega^\varepsilon(\zeta) \left(1 + |\langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle| + |\langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle| \sum_j |\langle \partial_\xi \psi_1^\varepsilon, \phi_j^\varepsilon \rangle| |w_j| \right. \right. \\ &\quad \left. \left. + \sum_j |\langle \partial_\xi \psi_k^\varepsilon, \phi_j^\varepsilon \rangle| |w_j| + \sum_j |\langle \psi_k^\varepsilon, \partial_\xi \phi_j^\varepsilon \rangle| |w_j| \sum_\ell |\langle \partial_\xi \psi_1^\varepsilon, \phi_\ell^\varepsilon \rangle| |w_\ell| \right) E_k(s, t) \right)^2 \\ &\leq C \left(\int_0^t \Omega^\varepsilon(\zeta) |w|_{L^2}^2 E_1(s, t) ds \right)^2 + C \sum_{k \geq 2} \left(\int_0^t \Omega^\varepsilon(\zeta) (1 + |w|_{L^2}^2) E_k(s, t) ds \right)^2 \end{aligned}$$

Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we infer

$$\begin{aligned} |w - z|_{L^2} &\leq C \int_0^t \Omega^\varepsilon(\zeta) |w|_{L^2}^2 E_1(s, t) ds + C \sum_{k \geq 2} \int_0^t \Omega^\varepsilon(\zeta) (1 + |w|_{L^2}^2) E_k(s, t) ds \\ &\leq C \int_0^t \Omega^\varepsilon(\zeta) \left\{ |w|_{L^2}^2 E_1(s, t) + (1 + |w|_{L^2}^2) \sum_{k \geq 2} E_k(s, t) \right\} ds. \end{aligned}$$

The assumption on the asymptotic behavior of the eigenvalues λ_k can now be used to bound the series. Indeed, there holds

$$\sum_{k \geq 2} E_k(s, t) \leq E_2(s, t) \sum_{k \geq 2} \frac{E_k(s, t)}{E_2(s, t)} \leq C (t - s)^{-1/2} E_2(s, t)$$

As a consequence, for unknown w such that $|w|_{L^2} \leq M$ for some $M > 0$, we infer

$$\begin{aligned} E_1(t, 0) |w - z|_{L^2} &\leq C \int_0^t \Omega^\varepsilon(\zeta) \left\{ |w - z|_{L^2}^2 E_1(s, 0) \right. \\ &\quad \left. + |z|_{L^2}^2 E_1(s, 0) + (t - s)^{-1/2} E_2(s, t) E_1(t, 0) \right\} ds \end{aligned}$$

Let us set

$$N(t) := \sup_{s \in [0, t]} |w - z|_{L^2} E_1(s, 0)$$

Then, since $\Lambda_2^\varepsilon \leq \Lambda_1^\varepsilon$, we obtain

$$\begin{aligned} E_1(t, 0) |w - z|_{L^2} &\leq C \int_0^t \Omega^\varepsilon(\zeta) N^2(s) E_1(0, s) ds \\ &\quad + C \int_0^t \Omega^\varepsilon(\zeta) \left\{ |w_0|_{L^2}^2 e^{(2\Lambda_2^\varepsilon - \Lambda_1^\varepsilon)s} + (t - s)^{-1/2} E_2(s, t) E_1(t, 0) \right\} ds \end{aligned}$$

By assumption (3.6), $\lambda_1^\varepsilon \leq -C\Omega^\varepsilon$ for some constant $C > 0$, hence

$$\begin{aligned} \int_0^t \Omega^\varepsilon(\zeta) N^2(s) E_1(0, s) ds &\leq \int_0^t \Omega^\varepsilon(\zeta) N^2(s) \exp \left(-C \int_0^s \Omega^\varepsilon(\zeta) d\sigma \right) ds \\ &\leq N^2(t) \left\{ 1 - \exp \left(-C \int_0^t \Omega^\varepsilon(\zeta) d\sigma \right) \right\}. \end{aligned}$$

Moreover, there holds

$$\begin{aligned} \int_0^t e^{(2\Lambda_2^\varepsilon - \Lambda_1^\varepsilon)s} ds &\leq \int_0^t e^{\Lambda_2^\varepsilon s} ds = \frac{1}{\Lambda_2^\varepsilon} (e^{\Lambda_2^\varepsilon t} - 1) \leq \frac{1}{|\Lambda_2^\varepsilon|} \\ \int_0^t (t - s)^{-1/2} E_2(s, t) ds &\leq \int_0^t (t - s)^{-1/2} e^{\Lambda_2^\varepsilon(t-s)} ds \leq \frac{1}{|\Lambda_2^\varepsilon|^{1/2}} \end{aligned}$$

so that, recalling that Λ_2^ε is bounded away from 0,

$$E_1(t, 0) |w - z|_{L^2} \leq C \left\{ N^2(t) \left[1 - \exp \left(-C \int_0^t \Omega^\varepsilon(\zeta) d\sigma \right) \right] + C |\Omega^\varepsilon|_\infty \left(|w_0|_{L^2}^2 + E_1(t, 0) \right) \right\}$$

Next, we end up with the estimate

$$N(t) \leq AN^2(t) + B \quad \text{with} \quad \begin{cases} A := C \left\{ 1 - \exp \left(-C \int_0^t \Omega^\varepsilon(\zeta) d\sigma \right) \right\}, \\ B := C |\Omega^\varepsilon|_{L^\infty} \left(|w_0|_{L^2}^2 + E_1(t, 0) \right) \end{cases}$$

Hence, as soon as

$$(3.11) \quad 4AB = 4C^2 |\Omega^\varepsilon|_{L^\infty} \left(|w_0|_{L^2}^2 + E_1(t, 0) \right) \left(1 - \exp \left(-C \int_0^t \Omega^\varepsilon(\zeta) d\sigma \right) \right) < 1$$

there holds

$$N(t) \leq \frac{2B}{1 + \sqrt{4AB}} \leq 2B = C |\Omega^\varepsilon|_{L^\infty} \left(|w_0|_{L^2}^2 + E_1(t, 0) \right)$$

that means, in term of the difference $w - z$,

$$|w - z|_{L^2} \leq C |\Omega^\varepsilon|_{L^\infty} \left(|w_0|_{L^2}^2 E_1(0, t) + 1 \right)$$

Condition (3.11) gives a constraint on the final time T^ε . Since $1 - e^{-C \int_0^t \Omega^\varepsilon(\zeta) d\sigma} < 1$, it is enough to ask

$$(3.12) \quad 4C^2 |\Omega^\varepsilon|_{L^\infty} \left(|w_0|_{L^2}^2 + E_1(t, 0) \right) < 1$$

to assure condition (3.11) is satisfied. Constraint (3.12) can be rewritten as

$$C \exp \left(- \int_0^t \Omega^\varepsilon(\zeta) d\sigma \right) \leq \exp \left(- \int_0^t \lambda_1^\varepsilon(\zeta) d\sigma \right) = E_1(t, 0) \leq \frac{C}{|\Omega^\varepsilon|_{L^\infty}} - |w_0|_{L^2}^2,$$

so that we can choose T^ε of the form

$$T^\varepsilon := \frac{1}{|\Omega^\varepsilon|_{L^\infty}} \ln \left(\frac{C}{|\Omega^\varepsilon|_{L^\infty}} - |w_0|_{L^2}^2 \right) \sim -C |\Omega^\varepsilon|_{L^\infty}^{-1} \ln |\Omega^\varepsilon|_{L^\infty}$$

for w_0 sufficiently small. □

As a consequence of the estimate (3.7), for $|w|_{L^2} < M$ for some $M > 0$, the function ζ satisfies

$$\frac{d\zeta}{dt} = \theta^\varepsilon(\zeta)(1 + r) \quad \text{with} \quad |r| \leq C(|w_0|_{L^2} e^{\Lambda_{\frac{\varepsilon}{2}} t} + |\Omega^\varepsilon|_{L^\infty}).$$

where the constant C depends also on M . In particular, if ε and $|w_0|_{L^2}$ are sufficiently small, the function $\zeta = \zeta(t)$ has similar decay properties with respect to the function η , solution to the reduced Cauchy problem

$$\frac{d\eta}{dt} = \theta^\varepsilon(\eta), \quad \eta(0) = \zeta_0.$$

This preludes to the following consequence of Theorem 3.1.

Corollary 3.3. *Let hypotheses **H1-2-3** and (3.6) be satisfied. Assume also*

$$(3.13) \quad s \theta^\varepsilon(s) < 0 \quad \text{for any } s \in I, s \neq 0 \quad \text{and} \quad \theta^{\varepsilon'}(\bar{\zeta}) < 0.$$

Then, for ε and $|w_0|_{L^2}$ sufficiently small, the estimate (3.7) holds globally in time and the solution (ζ, w) converges to $(\bar{\zeta}, 0)$ as $t \rightarrow +\infty$.

Proof. Thanks to assumption **H1**, for ε and $|w_0|_{L^2}$ sufficiently small, estimate (3.7) holds. Hence, for any initial datum ζ_0 , the variable $\zeta = \zeta(t)$ solves an equation of the form

$$\frac{d\zeta}{dt} = \theta^\varepsilon(\zeta)(1 + r(t)) \quad \text{with} \quad |r(t)| \leq C(|w_0|_{L^2} e^{\Lambda_2^\varepsilon t} + |\Omega^\varepsilon|_{L^\infty}).$$

Therefore, $\zeta(t)$ converges to $\bar{\zeta}$ as $t \rightarrow +\infty$ and the convergence is exponential, in the sense that there exists $\beta^\varepsilon < 0$ such that

$$(3.14) \quad |\zeta(t) - \bar{\zeta}| \leq |\zeta_0| e^{\beta^\varepsilon t}, \quad \beta^\varepsilon \sim \theta^{\varepsilon'}(\bar{\zeta})$$

for any t under consideration. Furthermore, from (3.9), we deduce

$$w_k(t) = w_k(0) \exp\left(\int_0^t \lambda_k^\varepsilon d\sigma\right) + \int_0^t \langle \psi_k^\varepsilon, F \rangle(s) \exp\left(\int_s^t \lambda_k^\varepsilon d\sigma\right) ds$$

Thus, by the Jensen's inequality, we infer the estimate

$$\begin{aligned} |w|_{L^2}^2(t) &\leq C \left\{ |w_0|_{L^2}^2 e^{2\Lambda_1^\varepsilon t} + \sum_k \left(\int_0^t \langle \psi_k^\varepsilon, F \rangle(s) e^{\Lambda_1^\varepsilon(t-s)} ds \right)^2 \right\} \\ &\leq C \left\{ |w_0|_{L^2}^2 e^{2\Lambda_1^\varepsilon t} + t \int_0^t |F|_{L^2}^2(s) e^{2\Lambda_1^\varepsilon(t-s)} ds \right\} \end{aligned}$$

Let $\nu^\varepsilon > 0$ be such that $|F|_{L^2}(t) \leq C e^{-\nu^\varepsilon t}$; then, if $\nu^\varepsilon \neq |\Lambda_1^\varepsilon|$, there holds

$$|w|_{L^2}^2(t) \leq C \left\{ |w_0|_{L^2}^2 e^{2\Lambda_1^\varepsilon t} + t (e^{-2\nu^\varepsilon t} + e^{2\Lambda_1^\varepsilon t}) \right\}$$

showing the exponential convergence to 0 of the component w . \square

Estimate (3.14) shows the exponentially slow motion of the shock layer for small ε . Precisely, the evolution of the location of the shock towards the equilibrium position is much slower as ε becomes smaller, since $\beta^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. For example, when $f(s) = s^2/2$, $\bar{\zeta} = 0$ and $\theta^{\varepsilon'}(0) \sim e^{-1/\varepsilon}$ (see formula (2.13)).

Let us also stress that in the regime $(\zeta, w) \sim (\bar{\zeta}, 0)$, a linearization at the equilibrium solution $U^\varepsilon(x; \bar{\zeta})$ would furnish a more detailed description of the dynamics, since the source term due to the approximation at an approximate steady state would not be present. In fact, the description given by the quasi-linearization is meaningful in the regime far from equilibrium and its aim is to describe the slow motion around a manifold of approximate solutions.

4. SPECTRAL ANALYSIS FOR THE DIFFUSION-TRANSPORT OPERATOR

Our concern in the present Section is to establish a precise description on the location of the eigenvalues of the linearized operator, in order to show that the general procedure developed in the previous Sections is indeed applicable in the case of Burgers equation. The problem of determining the limiting structure of the spectrum of the type of second order differential operators we deal with has been widely considered in the literature. Among others, let us quote the approach, based on the use of Prüfer transform, used in [5], in the context of metastability analysis for the Allen–Cahn equation. Here, we prefer to follow the strategy implemented in [14], for the linearization at the steady state of the Burgers equation. In what follows, we show that the same kind of eigenvalues distribution holds in a much more general situation, the main ingredient being the resemblance of the coefficient a^ε to a step function a^0 , jumping from a positive to a negative value, as $\varepsilon \rightarrow 0^+$.

Fixed $\varepsilon > 0$ and linearizing the scalar conservation law (2.2) at a given a reference profile $U^\varepsilon = U^\varepsilon(x; \xi)$, satisfying the boundary conditions $U^\varepsilon(\pm\ell; \xi) = u_\pm$, we end up with the differential linear diffusion-transport operator

$$(4.1) \quad \mathcal{L}_\xi^\varepsilon u := \varepsilon \partial_x^2 u - \partial_x(a^\varepsilon u) \quad u(\pm\ell) = 0,$$

where $a^\varepsilon = a^\varepsilon(x; \xi) := f'(U^\varepsilon(x; \xi))$. The aim of this Section is to describe the structure of the spectrum $\sigma(\mathcal{L}_\xi^\varepsilon)$ of the operator $\mathcal{L}_\xi^\varepsilon$ for ε sufficiently small.

Given the function a^ε , let us introduce the self-adjoint operator

$$(4.2) \quad \mathcal{M}_\xi^\varepsilon v := \varepsilon^2 \partial_x^2 v - b^\varepsilon v \quad v(\pm\ell) = 0,$$

where

$$(4.3) \quad b^\varepsilon := \left(\frac{1}{2} a^\varepsilon \right)^2 + \frac{1}{2} \varepsilon \frac{da^\varepsilon}{dx}.$$

By omitting the dependencies from ξ for shortness, a straightforward calculation shows that if u is an eigenfunction of (4.1) relative to the eigenvalue λ , then the function $v(x)$ defined by

$$v(x) = \exp \left(-\frac{1}{2\varepsilon} \int_{x_0}^x a^\varepsilon(y) dy \right) u(x)$$

(with x_0 arbitrarily chosen) is an eigenfunction of the operator $\mathcal{M}_\xi^\varepsilon$ relative to the eigenvalue $\mu := \varepsilon\lambda$. Since $\mathcal{M}_\xi^\varepsilon$ is self-adjoint, we can state that the spectrum of the operator $\mathcal{L}_\xi^\varepsilon$ is composed by real eigenvalues. Moreover, if u is an eigenfunction of (4.1) relative to the first eigenvalue λ_1^ε , integrating in $(-\ell, \ell)$ the relation $\mathcal{L}_\xi^\varepsilon u = \lambda u$, we deduce the identity

$$0 = \int_{-\ell}^{\ell} (\mathcal{L}_\xi^\varepsilon - \lambda_1^\varepsilon) u dx = \varepsilon (\partial_x u(\ell) - \partial_x u(-\ell)) - \lambda_1^\varepsilon \int_{-\ell}^{\ell} u(x) dx$$

Assuming, without loss of generality, u to be strictly positive in $(-\ell, \ell)$ and normalized so that its integral in $(-\ell, \ell)$ is equal to 1, we get

$$\lambda_1^\varepsilon = \varepsilon (\partial_x u(\ell) - \partial_x u(-\ell)) < 0$$

Hence, for any choice of the function a^ε , there holds

$$\sigma(\mathcal{L}_\xi^\varepsilon) \subset (-\infty, 0).$$

Our next aim is to show that under appropriate assumption on the behavior of the family of functions a^ε as $\varepsilon \rightarrow 0^+$, it is possible to furnish a detailed representation of the eigenvalue distributions for small ε . Specifically, we are interested in coefficients a^ε behaving, in the limit $\varepsilon \rightarrow 0^+$ as a step function of the form

$$(4.4) \quad a^0(x) := \begin{cases} a_- & x \in (-\ell, \xi), \\ a_+ & x \in (\xi, \ell), \end{cases}$$

for some $\xi \in (-\ell, \ell)$ and $a_+ < 0 < a_-$. We will show that, under appropriate assumptions making precise in which sense a^ε “resemble” a^0 for ε small, the first eigenvalue λ_1^ε turns to be “very close” to 0 for ε small, and all of the others eigenvalues λ_k^ε , with $k \geq 2$, are such that $\varepsilon \lambda_k^\varepsilon = O(1)$ as $\varepsilon \rightarrow 0^+$.

Estimate from below for the first eigenvalue. We estimate the first eigenvalue μ_1^ε of the operator $\mathcal{M}_\xi^\varepsilon$ by means of the inequality

$$|\mu_1^\varepsilon| \leq \frac{|\mathcal{M}_\xi^\varepsilon \psi|_{L^2}}{|\psi|_{L^2}}.$$

for smooth test function ψ such that $\psi(\pm\ell) = 0$. Let us consider as test function $\psi^\varepsilon(x) := \psi_0^\varepsilon(x) - K^\varepsilon(x)$, where

$$\begin{aligned} \psi_0^\varepsilon(x) &:= \exp\left(\frac{1}{2\varepsilon} \int_\xi^x a^\varepsilon(y) dy\right), \\ K^\varepsilon(x) &:= \frac{1}{2\ell} \{\psi_0^\varepsilon(-\ell)(\ell - x) + \psi_0^\varepsilon(\ell)(\ell + x)\}. \end{aligned}$$

A direct calculation shows that $\mathcal{M}_\xi^\varepsilon \psi := b^\varepsilon K$ and, assuming the family b^ε to be uniformly bounded, we infer

$$|\mu_1^\varepsilon| \leq \frac{|b^\varepsilon K^\varepsilon|_{L^2}}{|\psi_0^\varepsilon - K^\varepsilon|_{L^2}} \leq C \frac{|K^\varepsilon|_{L^2}}{|\psi_0^\varepsilon|_{L^2} - |K^\varepsilon|_{L^2}} = \frac{C}{|K^\varepsilon|_{L^2}^{-1} |\psi_0^\varepsilon|_{L^2} - 1}$$

as soon as $|\psi_0^\varepsilon|_{L^2} > |K^\varepsilon|_{L^2}$.

The opposite case being similar, let us assume $\psi_0(-\ell) \geq \psi_0(\ell)$. From the definition of K^ε , it follows

$$|K^\varepsilon|_{L^2}^2 = \frac{2\ell}{3} \{\psi_0^2(\ell) + \psi_0(\ell)\psi_0(-\ell) + \psi_0^2(-\ell)\} \leq 2\ell \psi_0^2(-\ell).$$

Therefore, we deduce

$$|K^\varepsilon|_{L^2}^{-2} |\psi_0^\varepsilon|_{L^2}^2 \geq 2\ell \psi_0^{-2}(-\ell) \int_{-\ell}^{\ell} |\psi_0^\varepsilon(x)|^2 dx = 2\ell I^\varepsilon$$

where

$$I^\varepsilon := \int_{-\ell}^{\ell} \exp\left(\frac{1}{\varepsilon} \int_{-\ell}^x a^\varepsilon(y) dy\right) dx$$

Since a^ε converges to the step function a^0 as $\varepsilon \rightarrow 0^+$, it is natural to approximate the latter integral in term of the corresponding one for a^0 :

$$I^\varepsilon = \int_{-\ell}^{\ell} \exp\left(\frac{1}{\varepsilon} \int_{-\ell}^x (a^\varepsilon - a^0)(y) dy\right) \exp\left(\frac{1}{\varepsilon} \int_{-\ell}^x a^0(y) dy\right) dx \geq e^{-|a^\varepsilon - a^0|_{L^1}/\varepsilon} I^0.$$

Since, for ε small,

$$\begin{aligned} I^0 &= \int_{-\ell}^{\xi} e^{a_-(x+\ell)/\varepsilon} dx + e^{a_-(\xi+\ell)/\varepsilon} \int_{\xi}^{\ell} e^{a_+(x-\xi)/\varepsilon} dx \\ &= \varepsilon e^{a_-(\xi+\ell)/\varepsilon} \left\{ \frac{1}{a_-} (1 - e^{-a_-(\xi+\ell)/\varepsilon}) - \frac{1}{a_+} (1 - e^{a_+(\ell-\xi)/\varepsilon}) \right\} \sim \frac{[a]}{a_- a_+} \varepsilon e^{a_-(\xi+\ell)/\varepsilon}. \end{aligned}$$

the subsequent estimate holds

$$|K^\varepsilon|_{L^2}^{-2} |\psi_0^\varepsilon|_{L^2}^2 \geq 2\ell e^{-|a^\varepsilon - a^0|_{L^1}/\varepsilon} I^0 \geq C_1 e^{C_2/\varepsilon}.$$

whenever $|a^\varepsilon - a^0|_{L^1} \leq c_0 \varepsilon$ for some $c_0 > 0$. Thus, we deduce for the first eigenvalue μ_1^ε of the self-adjoint operator $\mathcal{M}_\xi^\varepsilon$ the estimate $|\mu_1^\varepsilon| \leq C_1 e^{C_2/\varepsilon}$ for some positive constant C_1, C_2 . As a consequence, since the spectrum $\sigma(\mathcal{L}_\xi^\varepsilon)$ coincides with $\varepsilon^{-1} \sigma(\mathcal{M}_\xi^\varepsilon)$, the next result holds.

Proposition 4.1. *Let a^ε be a family of functions satisfying the assumption:*

A0. *there exists $C > 0$, independent on $\varepsilon > 0$, such that*

$$|a^\varepsilon|_{L^\infty} + \varepsilon \left| \frac{da^\varepsilon}{dx} \right|_{L^\infty} \leq C$$

If there exists $\xi \in (-\ell, \ell)$, $a_+ < 0 < a_-$ and $C > 0$ for which $|a^\varepsilon - a^0|_{L^1} \leq C\varepsilon$, then there exist constants $C, c > 0$ such that $-C e^{-c/\varepsilon} \leq \lambda_1^\varepsilon < 0$.

Let us stress that the request $a_+ < 0 < a_-$ is essential, even if hided in the proof. If this is not the case, the term K^ε would not be small as $\varepsilon \rightarrow 0^+$ and its L^2 norm would not be bounded by the L^2 -norm of ψ_0^ε . In fact, the statement in Proposition 4.1 may not hold when a_\pm have the same sign, the easiest example being the case $a^\varepsilon \equiv a_+ = a_- > 0$.

The next Example gives a heuristic estimate for the first eigenvalue λ_1^ε .

Example 4.2. Given $-\alpha < 0 < \beta$ and $a_\pm \in \mathbb{R}$, let us set $I = (-\alpha, \beta)$, $[a] := a_+ - a_-$ and

$$a(x) = a_- \chi_{(-\alpha, 0)}(x) + a_+ \chi_{(0, \beta)}(x).$$

Given $\lambda^\varepsilon > 0$, let us look for function $u \in C(I)$, such that

$$\mathcal{L}u := \varepsilon u'' + (a(x)u)' + \lambda^\varepsilon u = 0, \quad u(-\alpha) = u(\beta) = 0$$

in the sense of distributions. Since $a' = [a]\delta_0$, this amounts in finding two functions u^\pm such that

$$\mathcal{L}_\pm u := \varepsilon u''_\pm + a_\pm u'_\pm + \lambda^\varepsilon u = 0, \quad u_-(-\alpha) = u_+(\beta) = 0$$

and the following transmission conditions are satisfied

$$u_+(0) - u_-(0) = 0 \quad \text{and} \quad \varepsilon (u'_+(0) - u'_-(0)) + [a] u_\pm(0) = 0.$$

The characteristic polinomial of \mathcal{L}_\pm is $p_\pm(\mu; \lambda^\varepsilon) := \varepsilon \mu^2 + a_\pm \mu + \lambda^\varepsilon$, with roots

$$\mu_\pm^\pm := \frac{-a_\pm \pm \Delta_\pm}{2\varepsilon}, \quad \mu_\pm^\pm := \frac{-a_\pm \pm \Delta_\pm}{2\varepsilon}, \quad \text{where } \Delta_\pm := \sqrt{(a_\pm)^2 - 4\varepsilon \lambda^\varepsilon}.$$

Assume $\lambda^\varepsilon < (a_\pm)^2/4\varepsilon$. Choosing u_\pm in the form

$$u_-(x) = A_-(e^{\mu_-^+(\alpha+x)} - e^{\mu_-^-(\alpha+x)}) \quad \text{and} \quad u_+(x) = A_+(e^{-\mu_+^+(\beta-x)} - e^{-\mu_+^-(\beta-x)}).$$

Setting $\theta_-^\pm := e^{\mu_-^\pm \alpha}$ and $\theta_+^\pm := e^{-\mu_+^\pm \beta}$, there holds

$$\begin{aligned} u_-(0) &= A_-(\theta_-^+ - \theta_-^-) & u'_-(0) &= A_-(\mu_-^+ \theta_-^+ - \mu_-^- \theta_-^-) \\ u_+(0) &= A_+(\theta_+^+ - \theta_+^-) & u'_+(0) &= A_+(\mu_+^+ \theta_+^+ - \mu_+^- \theta_+^-). \end{aligned}$$

Therefore, the transmission conditions take the form of a linear system in A_\pm

$$(4.5) \quad \begin{cases} (\theta_+^+ - \theta_+^-)A_+ - (\theta_-^+ - \theta_-^-)A_- = 0, \\ \left\{ (2\varepsilon \mu_+^+ + [a]) \theta_+^+ - (2\varepsilon \mu_+^- + [a]) \theta_+^- \right\} A_+ \\ \quad + \left\{ (-2\varepsilon \mu_-^+ + [a]) \theta_-^+ + (2\varepsilon \mu_-^- - [a]) \theta_-^- \right\} A_- = 0. \end{cases}$$

After some manipulation, the determinant $D = D(\lambda^\varepsilon, \varepsilon)$ of (4.5) can be rewritten as

$$D = ([a] + [\Delta]) \theta_+^+ \theta_-^+ - ([a] - \{\Delta\}) \theta_+^- \theta_-^+ - ([a] + \{\Delta\}) \theta_+^+ \theta_-^- + ([a] - [\Delta]) \theta_+^- \theta_-^-,$$

where $[\Delta] := \Delta_+ - \Delta_-$ and $\{\Delta\} := \Delta_+ + \Delta_-$.

Since $\sqrt{\kappa^2 - 4x} = |\kappa| - 2|\kappa|^{-1}x + o(x)$, for $\varepsilon \lambda^\varepsilon \rightarrow 0$, there hold

$$\begin{aligned} \varepsilon \ln(\theta_+^+ \theta_-^+) &= |a_-| \alpha + \left(\frac{\beta}{a_+} + \frac{\alpha}{a_-} \right) \varepsilon \lambda^\varepsilon + o(\varepsilon \lambda^\varepsilon), \\ \varepsilon \ln(\theta_+^- \theta_-^+) &= (a_+ \beta + |a_-| \alpha) - \left(\frac{\beta}{a_+} + \frac{\alpha}{|a_-|} \right) \varepsilon \lambda + o(\varepsilon \lambda), \\ \varepsilon \ln(\theta_+^+ \theta_-^-) &= \left(\frac{\beta}{a_+} + \frac{\alpha}{|a_-|} \right) \varepsilon \lambda^\varepsilon + o(\varepsilon \lambda^\varepsilon), \\ \varepsilon \ln(\theta_+^- \theta_-^-) &= a_+ \beta - \left(\frac{\beta}{a_+} + \frac{\alpha}{a_-} \right) \varepsilon \lambda^\varepsilon + o(\varepsilon \lambda^\varepsilon) \end{aligned}$$

$$\begin{aligned}\{\Delta\} &= \sqrt{(a_+)^2 - 4\varepsilon\lambda^\varepsilon} + \sqrt{(a_-)^2 - 4\varepsilon\lambda^\varepsilon} = [a] \left(1 + \frac{2\varepsilon\lambda^\varepsilon}{a_+ a_-}\right) + o(\varepsilon\lambda^\varepsilon) \\ [\Delta] &= \sqrt{(a_+)^2 - 4\varepsilon\lambda^\varepsilon} - \sqrt{(a_-)^2 - 4\varepsilon\lambda^\varepsilon} = \{a\} \left(1 - \frac{2\varepsilon\lambda^\varepsilon}{a_+ a_-}\right) + o(\varepsilon\lambda^\varepsilon)\end{aligned}$$

Hence, we infer

$$D \sim 2 \left(a_+ e^{|a_-|\alpha/\varepsilon} + \frac{\varepsilon\lambda^\varepsilon}{a_+ a_-} e^{(a_+ \beta + |a_-|\alpha)/\varepsilon} - a_- e^{a_+ \beta/\varepsilon} \right),$$

so that $D \sim 0$ for

$$\lambda^\varepsilon \sim -\frac{a_+ a_-}{\varepsilon [a]} (a_+ e^{-a_+ \beta/\varepsilon} - a_- e^{-|a_-|\alpha/\varepsilon})$$

For $a_\pm = \pm u_*$, $\alpha = \ell + \xi$ and $\beta = \ell - \xi$, the above expression becomes

$$\lambda^\varepsilon \sim \frac{u_*^2}{2\varepsilon} (e^{-u_*(\ell-\xi)/\varepsilon} + e^{-u_*(\ell+\xi)/\varepsilon}) = \frac{u_*^2}{\varepsilon} \cosh(u_* \xi/\varepsilon) e^{-u_* \ell/\varepsilon}$$

to be compared with the expression for Ω^ε obtained in Example 2.1

$$\Omega^\varepsilon \sim \left| \frac{2u_*^2}{\varepsilon} (e^{-u_*(\ell+\xi)/\varepsilon} - e^{-u_*(\ell-\xi)/\varepsilon}) \right| = \frac{4u_*^2}{\varepsilon} |\sinh(u_* \xi/\varepsilon)| e^{-u_* \ell/\varepsilon}$$

so that

$$\left| \frac{\Omega^\varepsilon}{\lambda^\varepsilon} \right| \sim 4 |\tanh(u_* \xi/\varepsilon)| \leq 4$$

Let us stress that this formula shows that hypothesis (3.6) is verified heuristically for Burgers type equations.

Estimate from above for the second eigenvalue. Controlling the location of the second (and subsequent) eigenvalue needs much more care and, also, a number of additional assumption on the limiting behavior of the function a^ε as $\varepsilon \rightarrow 0^+$. Precisely, we suppose $a^\varepsilon \in C^0([-\ell, \ell])$ satisfies the following hypotheses:

A1. the function a^ε is twice differentiable at any $x \neq \xi$ and

$$\frac{da^\varepsilon}{dx}, \frac{d^2a^\varepsilon}{dx^2} < 0 < a^\varepsilon \quad \text{in } (-\ell, \xi), \quad \text{and} \quad a^\varepsilon, \frac{da^\varepsilon}{dx} < 0 < \frac{d^2a^\varepsilon}{dx^2} \quad \text{in } (\xi, \ell),$$

A2. for any $C > 0$ there exists $c_0 > 0$ such that, for any x satisfying $|x - \xi| \geq c_0 \varepsilon$, there holds

$$|a^\varepsilon - a^0| \leq C \varepsilon \quad \text{and} \quad \varepsilon \left| \frac{da^\varepsilon}{dx} \right| \leq C;$$

A3. there exists the left/right first order derivatives of a^ε at ξ and

$$\liminf_{\varepsilon \rightarrow 0^+} \varepsilon \left| \frac{da^\varepsilon}{dx}(\xi \pm) \right| > 0$$

As a consequence, the function $b^\varepsilon + \varepsilon\lambda^\varepsilon$ satisfies a number of corresponding properties, listed in the next statement.

Lemma 4.3. *Let the family a^ε be such that hypotheses A1-2-3 are satisfied, and let $\lambda^\varepsilon < 0$ be such that*

$$(4.6) \quad \inf_{\varepsilon > 0} \varepsilon \lambda^\varepsilon > -\frac{1}{4} \alpha_0^2 \quad \text{where } \alpha_0 := \min\{|a_-|, |a_+|\}.$$

Then there exist $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, the functions $b^\varepsilon + \varepsilon \lambda^\varepsilon$, with b^ε defined in (4.3), enjoy the following properties:

- B1. *the function $b^\varepsilon + \varepsilon \lambda^\varepsilon$ is decreasing in $(-\ell, \xi)$ and increasing in (ξ, ℓ) ;*
- B2. *there exist $C, c > 0$ such that, for any x with $|x - \xi| \geq c\varepsilon$ there holds $b^\varepsilon + \varepsilon \lambda^\varepsilon \geq C > 0$;*
- B3. *there exist the left/right limits of $b^\varepsilon + \varepsilon \lambda^\varepsilon$ at ξ and*

$$\beta := \limsup_{\varepsilon \rightarrow 0^+} (b^\varepsilon(\xi \pm) + \varepsilon \lambda^\varepsilon) < 0;$$

Proof. Property B1. is an immediate consequence of assumption A1, since

$$\frac{d}{dx} (b^\varepsilon + \varepsilon \lambda^\varepsilon) = \frac{1}{4} a^\varepsilon \frac{da^\varepsilon}{dx} + \frac{1}{2} \varepsilon \frac{d^2 a^\varepsilon}{dx^2}.$$

From A2, given $C > 0$, for $x \leq \xi - c_0 \varepsilon$, there holds

$$\begin{aligned} b^\varepsilon + \varepsilon \lambda^\varepsilon &\geq \frac{1}{4} (a^\varepsilon + a^0)(a^\varepsilon - a^0) - \frac{1}{2} \varepsilon \left| \frac{da^\varepsilon}{dx} \right| + \varepsilon \lambda^\varepsilon + \frac{1}{4} a_-^2 \\ &\geq \varepsilon \lambda^\varepsilon + \frac{1}{4} \alpha_0^2 - \frac{1}{2} \left(1 + |a^0| \varepsilon + \frac{1}{2} C \varepsilon^2 \right) C \end{aligned}$$

From such inequality, by choosing $C > 0$ sufficiently small, and combining with an analogous estimate on $(\xi + c\varepsilon, \ell)$, property B2. follows.

For what concerns B3, we observe that, since $a(\xi) = 0$ and $\lambda \leq 0$, there holds

$$\limsup_{\varepsilon \rightarrow 0^+} (b^\varepsilon(\xi \pm) + \varepsilon \lambda^\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0^+} \frac{1}{2} \varepsilon \frac{da^\varepsilon}{dx}(\xi) = -\liminf_{\varepsilon \rightarrow 0^+} \varepsilon \left| \frac{da^\varepsilon}{dx}(\xi \pm) \right| < 0,$$

thanks to A3. □

For later reference, we denote y_\pm^ε the zeros of $b^\varepsilon + \varepsilon \lambda^\varepsilon$, with $-\ell < y_-^\varepsilon < \xi < y_+^\varepsilon < \ell$. Since property B2 holds, we deduce that $|y_\pm^\varepsilon - \xi| \leq c_0 \varepsilon$.

Assume the assumption of Lemma 4.3 to hold, and let λ_2^ε and $\mu_2^\varepsilon = \varepsilon \lambda_2^\varepsilon$ be the second eigenvalue of the operators $\mathcal{L}_\xi^\varepsilon$ and $\mathcal{M}_\xi^\varepsilon$, respectively, with corresponding eigenfunctions ϕ_2^ε and ψ_2^ε . Such eigenfunctions are linked together by the relation

$$(4.7) \quad \psi_2^\varepsilon(x) = A \exp \left(-\frac{1}{2\varepsilon} \int_{x_*}^x a^\varepsilon(y) dy \right) \phi_2^\varepsilon(x)$$

for some constants A and x_* . Since λ_2^ε is the second eigenvalue, the functions ϕ_2^ε and ψ_2^ε possess a single root located at some point $x_0^\varepsilon \in (-\ell, \ell)$. The sign properties of $b^\varepsilon + \mu_2^\varepsilon$ described in Lemma 4.3 imply that $x_0^\varepsilon \in (y_-^\varepsilon, y_+^\varepsilon)$. Then, ϕ_2^ε and ψ_2^ε restricted to the intervals $(-\ell, x_0^\varepsilon)$ and (x_0^ε, ℓ) are eigenfunctions relative to the first eigenvalue

of the same operator considered in the corresponding intervals and with Dirichlet boundary conditions.

From now on, we drop, for shortness, the dependence on ε of $\lambda_2, \phi_2, \psi_2, x_0$, we assume, without loss of generality, $x_0 \geq \xi$ and we restrict our attention to the interval $J = (x_0, \ell)$. Integrating on J , we deduce

$$\lambda_2 \int_{x_0}^{\ell} \phi_2 dx = \varepsilon (\partial_x \phi_2(\ell) - \partial_x \phi_2(x_0)) < -\varepsilon \partial_x \phi_2(x_0)$$

having chosen ϕ_2 positive in J . Assuming ψ_2 to be given as in (4.7) with $A = 1$ and $x_* = x_0$, and normalized so that $\max \psi_2 = 1$, from the latter inequality we infer the inequality

$$(4.8) \quad |\lambda_2| > \varepsilon I^{-1} \partial_x \psi_2(x_0),$$

where

$$I := \int_{x_0}^{\ell} \exp \left(\frac{1}{2\varepsilon} \int_{x_0}^x a^\varepsilon(y) dy \right) dx$$

Our next aim is to deduce an estimate from above on I_ε and an estimate from below for $\partial_x \psi_2(x_0)$, in order to get a control on the size of the second eigenvalue λ_2 .

From the definition of I_ε , since $x_0 \geq \xi$, it follows

$$\begin{aligned} I_\varepsilon &\leq e^{|a^\varepsilon - a^0|_{L^1}/2\varepsilon} \int_{x_0}^{\ell} e^{a_+(x-x_0)/2\varepsilon} dx = \frac{2\varepsilon}{|a_+|} e^{|a^\varepsilon - a^0|_{L^1}/2\varepsilon} (1 - e^{a_+(\ell-x_0)/2\varepsilon}) \\ &\leq \frac{2\varepsilon}{|a_+|} e^{|a^\varepsilon - a^0|_{L^1}/2\varepsilon} \leq C \varepsilon \end{aligned}$$

whenever $|a^\varepsilon - a^0|_{L^1} \leq C \varepsilon$. Thus, estimate (4.8) provisionally becomes

$$(4.9) \quad |\lambda_2| > C \frac{d\psi_2}{dx}(x_0)$$

for some positive constant C , independent on ε .

Let x_M be such that $\psi_2(x_M) = 1$, minimum with such property. From the assumption on $b^\varepsilon + \varepsilon \lambda$, it follows $x_M \in (x_0, y_+)$. Then there exists $x_L \in (x_0, x_M)$ such that

$$\frac{d\psi_2}{dx}(x_L) = \frac{1}{x_M - x_0} \geq \frac{1}{y_+ - \xi} \geq \frac{1}{c_0 \varepsilon}.$$

Since the function ψ is concave in the interval (x_0, y_+) , we deduce

$$\frac{d\psi_2}{dx}(x_0) \geq \frac{d\psi_2}{dx}(x_L) \geq \frac{1}{c_0 \varepsilon}.$$

Plugging into (4.9), we end up with

$$(4.10) \quad |\lambda_2| \geq \frac{C}{\varepsilon}.$$

for some C independent on ε .

As a consequence, we can state the next result relative to the second eigenvalue λ_2 .

Proposition 4.4. *Let a^ε be a family of functions satisfying A1-2-3 then there exists a constant $C > 0$ such that $\lambda_2^\varepsilon \leq -C/\varepsilon$ for any ε sufficiently small.*

Spectral estimates. Collecting the results of Propositions 4.1 and 4.4 give a complete description for the spectrum of operator $\mathcal{L}_\xi^\varepsilon$ for small ε , under assumptions A0-1-2-3 on the family of functions a^ε .

Corollary 4.5. *Let a^ε be a family of functions satisfying the assumptions A0-1-2-3 for some $\xi \in (-\ell, \ell)$, $a_+ < 0 < a_-$. Then there exist constants $c, C_1, C_2 > 0$ such that*

$$\lambda_k^\varepsilon \leq -C_1/\varepsilon \quad \text{and} \quad -C_2 e^{-c/\varepsilon} \leq \lambda_1^\varepsilon < 0.$$

for any $k \geq 2$.

Hypotheses A0-1-2-3 are satisfied in the case of a family of function a^ε that is a (small) perturbation of a function \bar{a}^ε with the form

$$\bar{a}^\varepsilon(x) = A_- \left(\frac{x - \xi}{\varepsilon} \right) \chi_{(-L, \xi)}(x) + A_+ \left(\frac{x - \xi}{\varepsilon} \right) \chi_{(\xi, L)}(x).$$

for some decreasing smooth bounded functions A_\pm , bounded together with their first and second order derivatives, and such that $A_\pm(\pm\infty) = a_\pm$ and $A'_\pm(\pm\infty) = 0$.

5. APPENDIX. THE HYPERBOLIC DYNAMICS

In this Section, we concentrate on the dynamics of the scalar conservation law

$$(5.1) \quad \partial_t u + \partial_x f(u) = 0$$

with $x \in (-\ell, \ell)$, together with boundary condition $u(\pm\ell, t) = u_\pm$, and a given initial datum $u(x, 0) = u_0(x)$. Our aim is to give a self-contained proof of the *finite-time stabilization* of the solution under appropriate assumption on the flux function f and on the boundary values u_\pm . This kind of properties has been showed for the first time in [18] in the case of the Cauchy problem.

Theorem 5.1. *Assume the function f to be uniformly convex, i.e. $f''(s) \geq c_0 > 0$ for some constant c_0 . If u_-, u_+ are such that $u_+ < 0 < u_-$ and $f(u_+) = f(u_-)$, then, for any $u_0 \in BV(-\ell, \ell)$, the solution u to the initial value problem (5.1), $u(\pm\ell, t) = u_\pm$, $u(x, 0) = u_0(x)$ is such that for some $T > 0$ and $\xi \in [-\ell, \ell]$, there holds*

$$u(\cdot, T) = U_{hyp}(\cdot; \xi) \quad \text{in } (-\ell, \ell)$$

where $U_{hyp}(x; \xi) := u_- \chi_{(-\ell, \xi)}(x) + u_+ \chi_{(\xi, \ell)}(x)$.

To prove the statement, we use the *theory of generalized characteristics*, introduced in [6]. The convexity assumption on the flux function f guarantees that for any point $(x, t) \in (-\ell, \ell) \times (0, +\infty)$ there exist a minimal, respectively maximal, backward characteristics, which are classical characteristic curves, hence a straight lines with slope $f'(u(x-0, t))$, resp. $f'(u(x+0, t))$.

The boundary conditions are understood in the sense of Bardos–leRoux–Nédélec [2], meaning that the trace of the solution at the boundary is requested to take values in appropriate sets. To be precise, let $u_* \in (u_+, u_-)$ be such that $f'(u_*) = 0$ and set

$$\mathcal{R}u := \begin{cases} w & \text{if } \exists w \neq u \text{ s.t. } f(w) = f(u), \\ u_* & \text{if } u = u_*, \end{cases}$$

Then, skipping the details (see [20]), the boundary conditions $u(\pm\ell, t) = u_{\pm}$ translate into

$$u(-\ell+0, t) \in (-\infty, \mathcal{R}u_-] \cup \{u_-\}, \quad u(\ell-0, t) \in \{u_+\} \cup [\mathcal{R}u_+, +\infty)$$

Since $f(u_+) = f(u_-)$, there holds $\mathcal{R}u_{\pm} = u_{\mp}$, and the condition can be rewritten as

$$(5.2) \quad u(-\ell+0, t) \in (-\infty, u_+] \cup \{u_-\}, \quad u(\ell-0, t) \in \{u_+\} \cup [u_-, +\infty)$$

In particular, characteristic curves entering in the domain from the left side $x = -\ell$ (respectively, from the right side $x = \ell$) possess speed $f'(u_-)$ (resp. speed $f'(u_+)$).

Now, we are ready to prove Theorem 5.1.

Proof. Let $u = u(x, t)$ be the solution to the initial-boundary value problem under consideration with initial datum u_0 . For later use, we set

$$\begin{aligned} \zeta_-(t) &:= \sup\{x \in [-\ell, \ell] : u(y, t) = 1 \quad \forall y \in (-\ell, x)\} \cup -\ell, \\ \zeta_+(t) &:= \inf\{x \in [-\ell, \ell] : u(y, t) = -1 \quad \forall y \in (x, \ell)\} \cup \ell. \end{aligned}$$

In particular, $\zeta_- \leq \zeta_+$. We are going to show that $\zeta_-(T) = \zeta_+(T)$ for some $T > 0$.

1. *There exists $T_0 > 0$ such that $u(x, t) \in [u_+, u_-]$ for any $x \in (-\ell, \ell)$.*

Indeed, let $\bar{u} = \bar{u}(x, t)$ be the solution to the Riemann problem for (5.1) with initial datum

$$\bar{u}_0(x) = \begin{cases} u_- & x < -\ell, \\ \max\{u_-, \sup u_0\} & x > -\ell, \end{cases}$$

Hence, the restriction of \bar{u} to $(-\ell, \ell) \times (0, \infty)$ is a super-solution to the initial boundary value problem under consideration and, by comparison principle for entropy solution, we infer $u(x, t) \leq \bar{u}(x, t)$. Since $\bar{u}(x, t) = u_-$ for any $x < f'(u_-)t - \ell$, there holds

$$u(x, t) \leq u_- \quad \text{for } x \in (-\ell, \ell), \quad t \geq 2\ell/f'(u_-).$$

A similar estimate from below can be obtained by considering as subsolution the restriction of \underline{u} to $(-\ell, \ell) \times (0, \infty)$, where \underline{u} is the solution to (5.1) with initial datum

$$\underline{u}_0(x) = \begin{cases} \min\{u_+, \inf u_0\} & x < \ell, \\ u_+ & x > \ell, \end{cases}$$

From now on, we assume that the solution u takes values in the interval $[u_-, u_+]$.

2. Assume that $-\ell < \zeta_-(t) \leq \zeta_+(t) < \ell$ for any t ; then there exists $T_1 > 0$ such that $u(\zeta_-(t) + 0, t) < u_-$ and $u_+ < u(\zeta_+(t) - 0, t)$ for any $t > T_1$.

If u is continuous at $(\zeta_-(\tau), \tau)$ for some $\tau > 0$, then $u(\zeta_-(\tau) + 0, t) = u_-$. Therefore, the maximal backward characteristic from $(\zeta_-(\tau), \tau)$ is the straight line $x = \zeta_-(\tau) + f'(u_-)(t - \tau)$. For $\tau > 2L/f'(u_-)$, such curve intersects the boundary $x = -\ell$ at some $\sigma \in (0, \tau)$. By continuity, all of the maximal backward characteristics from (ξ, τ) with $\xi > \zeta_-(t)$ and sufficiently close to $\zeta_-(\tau)$ intersect the boundary $x = -\ell$ at some time $\sigma_*(\xi)$ smaller than σ and close to it. Because of the boundary conditions, this may happen if and only if $u(\xi, \tau) = u_-$. Hence, $u(x, \tau) = u_-$ for $x \in (\zeta_-(\tau), \zeta_-(\tau) + \varepsilon)$ for some $\varepsilon > 0$, in contradiction with the definition of ζ_- . Thus, continuity of u at $(\zeta_-(\tau), \tau)$ may happen only for $\tau \leq 2L/f'(u_-)$. A similar assertion holds for ζ_+ .

3. There exist $T > 0$ and $\xi \in [-\ell, \ell]$ such that $u(x, t) = U_{\text{hyp}}(\cdot; \xi)$ for any $t \geq T$.

Given $\theta > 0$, let $T_\theta := 2\ell/\theta$ be such that

$$u_-^\theta := u(\zeta_-(T_\theta) + 0, T_\theta) < u_- \quad \text{and} \quad u_+ < u_-^\theta := u(\zeta_+(T_\theta) - 0, T_\theta).$$

Let x_-^θ be the maximal backward characteristic from $(\zeta_-(T_\theta), T_\theta)$, whose equation is $x = \zeta_-(T_\theta) + f'(u_-^\theta)(t - T_\theta)$. If x_-^θ hits the right boundary $x = \ell$ at some positive time, the solution u coincides with $U_{\text{hyp}}(x; \zeta_-(T_\theta))$. Otherwise, there holds $\zeta_-(T_\theta) - f'(u_-^\theta)T_\theta < \ell$, which gives

$$f'(u_-^\theta) > \frac{\zeta_-(T_\theta) - \ell}{T_\theta} \geq -\frac{2\ell}{T_\theta} = -\theta$$

Similarly, let x_+^θ be the maximal backward characteristic from $(\zeta_+(T_\theta), T_\theta)$, whose equation is $x = \zeta_+(T_\theta) + f'(u_+^\theta)(t - T_\theta)$. If x_+^θ does not intersect the left boundary $x = -\ell$ at some positive time, there holds $f'(u_+^\theta) < \theta$.

Hence, for any $\varepsilon > 0$, we can choose θ sufficiently large so that $u_-^\theta > u_* - \varepsilon$ and $u_+^\theta < u_* + \varepsilon$. Thus, we have

$$\frac{d\zeta_+}{dt} - \frac{d\zeta_-}{dt} < \frac{f(u_+) - f(u_* + \varepsilon)}{u_+ - u_* - \varepsilon} - \frac{f(u_-) - f(u_* - \varepsilon)}{u_- - u_* + \varepsilon}$$

which is uniformly negative for ε sufficiently small. Hence, the curves ζ_+ and ζ_- intersect at some finite positive time $T > 0$. \square

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